

Preprint OCU-157, hep-th/9510172

# Differential Geometry and Integrability of the Hamiltonian System of a Closed Vortex Filament\*

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October 1995 (revised, March 1996)

**Abstract.** The system of a closed vortex filament is an integrable Hamiltonian one, namely, a Hamiltonian system with an infinite sequence of constants of motion in involution. An algebraic framework is given for the aim of describing differential geometry of this system. A geometrical structure related to the integrability of this system is revealed. It is not a bi-Hamiltonian structure but similar one. As a related topic, a remark on the inspection of J. Langer and R. Perline, J. Nonlinear Sci. 1, 71 (1991), is given.

**Mathematics Subject Classification (1991).** 58F07, 76C05.

## 1 Introduction

The vortex filament equation, which arises as the equation of motion for the vorticity in a certain approximation of 3-dimensional fluid forming a filament [1], has a description as an infinite dimensional nonlinear Hamiltonian system [2]. This is an integrable system in the sense that there exists an infinite sequence of constants of motion in involution,

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\*Title has been changed (previous title: Differential Geometry and Integrability of the Hamiltonian System of Closed Vortex Filament).

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namely, Poisson-commutative functions including the Hamiltonian function [3]. To assert this statement rigorously, some boundary condition or restriction on the configuration of filament should be imposed relevantly.

As a relevant phase space of the filament or Poisson manifold for this system, J. Langer and R. Perline [3] introduced *BAL*, which consists of the points represented by a single curve (position of the vortex filament) with infinite length, non-vanishing curvature and asymptotic convergence to a fixed line. They studied the Poisson structure of *BAL* and defined the sequense of functions (or functionals)  $I_{-2}, I_{-1}, \dots$  in involution by means of relating, via the Hasimoto map [4], the system of vortex filament to that of the nonlinear Schrödinger (NLS) equation [5], a well-known and well-investigated integrable Hamiltonian system. The Hamiltonian system of NLS equation has a bi-Hamiltonian structure, which is known as a geometrical background supporting the integrability [6, 7]. Although it is not quite obvious, the connection of the filament of *BAL* to the NLS seems to suggest that *BAL* also has a bi-Hamiltonian structure.

Let  $\kappa$ ,  $\tau$  and  $s$  denote the curvature, torsion and arclength, respectively, along the curve, *cf.* eq. (4). Let  $\kappa^{(i)} := (\frac{\partial}{\partial s})^i \kappa$  and  $\tau^{(i)} := (\frac{\partial}{\partial s})^i \tau$ . We refer to a polynomial in  $\kappa^{(i)}$  and  $\tau^{(i)}$ ,  $i = 0, 1, \dots$ , with coefficients in  $\mathbf{R}$  as a local polynomial. We note that the functions  $I_{n-2}$  were expressed as  $I_{n-2} = \int P_n ds$  with the local polynomials  $P_n$ ,

$$P_0 = 1, \quad P_1 = -\tau, \quad P_2 = \frac{1}{2} \kappa^2, \quad P_3 = \frac{1}{2} \kappa^2 \tau, \quad P_4 = \frac{1}{2} \kappa^{(1)2} + \frac{1}{2} \kappa^2 \tau^2 - \frac{1}{8} \kappa^4, \quad \dots, \quad (1)$$

see [3] for the definition.

In this paper we study the vortex filament with another restriction, that is, to form a closed curve. As a phase space or Poisson manifold for this system, we will introduce *ECL* (space of Entirely Curved Loop).

The analysis parallel to that of J. Langer and R. Perline does not work for *ECL*, because in this case the image (in the space of solutions to the NLS equation) of the Hasimoto map spans a space in which the NLS system has not been well-investigated, that is, the space of pseudoperiodic  $\mathbf{C}$ -valued functions with various (not fixed) periods and phase-displacements around a period.

One can, however, observe the existence of infinite sequense of functions on *ECL* in involution. Since every equation for local polynomials makes sense locally on the curve without reflection of the boundary condition, the easiest way to see this is to translate results on *BAL* regarding local polynomials into statements on *ECL*, *cf.* Sect. 5. It certainly explains the integrability of *ECL*; nevertheless the consideration within local polynomials seems awkward to clarify the structure lying behind the integrability. Indeed, for both systems *BAL* and *ECL*, a recursive formula associated with the sequense of functions in

involution exists and is written in a simple form by adopting indefinite integration, whose action cannot close within the local polynomials.

Our investigation about *ECL* in order to elucidate a geometrical structure related to the integrability is therefore adopting indefinite integration and is made entirely within *ECL* itself. Sect. 5 is exceptional; a remark related to *BAL* is given there. Although the space of dimension 3, in which the vortex filament live, is originally Euclidean, remarkable properties hold for the vortex filament in the space of constant curvature [8, 9]. We follow this generalization.

The paper is organized as follows: Sect. 2 is devoted to giving an algebraic framework, which describes periodic functions in an abstract manner in connection with integro-differential operators. With the aid of this framework, we introduce in Sect. 3 algebraic objects that allow one to define the differential calculus on *ECL*. A geometrical structure of *ECL* related to the integrability is elucidated in Sect. 4. This structure is related also to the inspection reported in [3] regarding the constants of motion for *BAL*. This is argued in Sect. 5.

## 2 Abstract Periodic Functions

We introduce with an axiomatic definition the notion of  $\partial^{\mathbf{Z}}$ -algebra, an abstraction of the algebra of commutative-algebra-valued periodic functions.

We say the set  $(A, \{\partial^m\}_{m \in \mathbf{Z}})$  of a unital, commutative and associative  $\mathbf{R}$ -algebra  $A$  and a sequence of  $\mathbf{R}$ -linear operators  $\partial^m: A \rightarrow A$  is a  $\partial^{\mathbf{Z}}$ -algebra if

$$\partial^m \partial^n f = \partial^{m+n} f, \quad \partial^1(fg) = (\partial^1 f)g + f(\partial^1 g), \quad \partial^{-1}((f - \partial^0 f)g) = (f - \partial^0 f)(\partial^{-1} g) \quad (2)$$

for every  $f, g \in A$ . Below, we refer simply to such an algebra  $A$  as a  $\partial^{\mathbf{Z}}$ -algebra. Definite integration in a  $\partial^{\mathbf{Z}}$ -algebra is defined by  $\int := \text{id} - \partial^0$ .

**EXAMPLE 1** *The algebra of smooth functions on  $S^1 := \mathbf{R}/\mathbf{Z}$  has a  $\partial^{\mathbf{Z}}$ -algebra structure. Let  $f$  be such a function, i.e.,  $f(x+1) = f(x) \forall x \in \mathbf{R}$ . Then  $\partial^1$  is realized by  $(\partial^1 f)(x) = df(x)/dx$ . And  $\partial^{-1}$  is realized by  $\partial^{-1} f = \tilde{\partial}^0 \tilde{\partial}^{-1} \tilde{\partial}^0 f$ , where  $(\tilde{\partial}^0 f)(x) = f(x) - \int_0^1 f(y) dy$  and  $(\tilde{\partial}^{-1} f)(x) = \int_0^x f(y) dy$ .*

Suppose  $A$  is a  $\partial^{\mathbf{Z}}$ -algebra. Then  $F := \int(A)$  is shown to be a  $\partial^{\mathbf{Z}}$ -subalgebra of  $A$ . The operators  $\partial^m$  act in  $F$  in the trivial way;  $\partial^m f = 0$ ,  $\partial^m(fg) = f(\partial^m g) \forall m \in \mathbf{Z}, \forall f \in F, \forall g \in A$ . The formula  $\int((\partial^{\pm 1} f)g + f(\partial^{\pm 1} g)) = 0 \forall f, g \in A$  is often useful. Note that the map  $\int: A \rightarrow F$  is an (end)morphism of  $\mathbf{R}$ -vector spaces, and not of algebras.

We say  $I$  is an ideal of  $\partial^{\mathbf{Z}}$ -algebra  $A$  if  $I$  is a not-unital  $\partial^{\mathbf{Z}}$ -subalgebra of  $\partial^{\mathbf{Z}}$ -algebra  $A$  and is an ideal (as algebra) of algebra  $A$ . Suppose this situation and let  $F := f(A)$ . Then,  $f(I)$  coincides with  $I \cap F$  and is an ideal of  $\partial^{\mathbf{Z}}$ -algebra  $F$ . The quotient algebra  $F/(I \cap F)$  can be considered a subalgebra of  $A/I$  and is identical to the algebra  $f(A/I)$ . Note that the quotient space  $A/I$  admits a natural  $\partial^{\mathbf{Z}}$ -algebra structure induced from  $A$ , so that  $\partial^m$  and  $f$  make sense as operators on  $A/I$ .

**EXAMPLE 2** Let  $A$  be a  $\partial^{\mathbf{Z}}$ -algebra. Then, the subset  $I := \{f \in A \mid \int fg = 0 \ \forall g \in A\}$  of  $A$  is an ideal of  $\partial^{\mathbf{Z}}$ -algebra  $A$ .

### 3 Formal Closed Curves

What we wish to describe is an unparametrized closed curve, but we start with considering parametrized one, which is more convenient for the variational calculus. The space where the curve lives is supposed to be a 3-dimensional orientable Riemannian manifold  $(M, \langle \cdot, \cdot \rangle)$  of constant curvature; the curvature of  $M$  is denoted by  $c \in \mathbf{R}$ , which is a given constant for the theory below. The reader who is interested only in the case that  $M$  is Euclidean can simply set  $c = 0$  throughout the paper.

The curve can almost be specified by the curvature  $\kappa$  and torsion  $\tau$  of the curve and the velocity  $\lambda$  associated with the parametrization. We will develop the theory based on these quantities. This means 6 degrees of freedom, *i.e.*, gross position and direction of the curve, are neglected at the starting point, but this seems relevant for discussing the integrability of the system of a single vortex filament. We suppose  $\lambda$  and  $\kappa$  are non-vanishing at every point on the curve.

We mention that we treat  $\lambda$ ,  $\kappa$  and  $\tau$  formally with forgetting their character of maps associating a function on  $S^1$  with a curve as I. M. Gel'fand and L. A. Dikii have done in their formal variational calculus [6, 10];  $\lambda$ ,  $\kappa$ ,  $\tau$  and their derivatives are read simply as indeterminates. There are differences in several points between our calculus and that of [6, 10]. Among them, here we stress that our calculus is based on the framework of  $\partial^{\mathbf{Z}}$ -algebra so that the formal curve is implicitly of a closed shape.

Let  $\overline{\mathcal{A}}_0$  be the unital, commutative and associative  $\mathbf{R}$ -algebra generated by the indeterminates  $\lambda^{(i)}$ ,  $\kappa^{(i)}$ ,  $\tau^{(i)}$  ( $i = 0, 1, \dots$ ),  $\lambda^{-1}$  and  $\kappa^{-1}$  satisfying  $\lambda^{-1}\lambda^{(0)} = \kappa^{-1}\kappa^{(0)} = 1$ . We denote  $\lambda := \lambda^{(0)}$ ,  $\kappa := \kappa^{(0)}$  and  $\tau := \tau^{(0)}$ . The  $\mathbf{R}$ -linear map  $\frac{\partial}{\partial s}: \overline{\mathcal{A}}_0 \rightarrow \overline{\mathcal{A}}_0$  is defined by  $\frac{\partial}{\partial s}\lambda^{(i)} = \lambda^{(i+1)}$ ,  $\frac{\partial}{\partial s}\kappa^{(i)} = \kappa^{(i+1)}$ ,  $\frac{\partial}{\partial s}\tau^{(i)} = \tau^{(i+1)}$  and Leibnitz rule  $\frac{\partial}{\partial s}(fg) = (\frac{\partial}{\partial s}f)g + f(\frac{\partial}{\partial s}g)$   $\forall f, g \in \overline{\mathcal{A}}_0$ .

Identifying  $\partial^1 = \lambda \frac{\partial}{\partial s}$ , we introduce  $\overline{\mathcal{A}}$  as the minimal extension of  $\overline{\mathcal{A}}_0$  such that  $\overline{\mathcal{A}}$  is a

$\partial^{\mathbf{Z}}$ -algebra. Let  $\overline{\mathcal{F}} := \mathcal{F}(\overline{\mathcal{A}})$  and  $\overline{\mathcal{X}} := \overline{\mathcal{A}}\mathbf{T} \oplus \overline{\mathcal{A}}\mathbf{N} \oplus \overline{\mathcal{A}}\mathbf{B}$ . Here, the set of  $\mathbf{T}$ ,  $\mathbf{N}$  and  $\mathbf{B}$  stands for a certain basis of  $\overline{\mathcal{X}}$  orthonormal with respect to the metric  $\langle \cdot, \cdot \rangle: \overline{\mathcal{X}} \times \overline{\mathcal{X}} \rightarrow \overline{\mathcal{A}}$ . The metric  $\langle \cdot, \cdot \rangle$  is supposed  $\overline{\mathcal{A}}$ -linear in each argument.

Let us define  $\mathbf{R}$ -linear operator  $\nabla_s: \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$  by Leibnitz rule and Frenet-Serret relation

$$\nabla_s(f\mathbf{X}) = (\frac{\partial}{\partial s}f)\mathbf{X} + f(\nabla_s\mathbf{X}) \quad \forall f \in \overline{\mathcal{A}}, \forall \mathbf{X} \in \overline{\mathcal{X}}, \quad (3)$$

$$\nabla_s\mathbf{T} = \kappa\mathbf{N}, \quad \nabla_s\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \nabla_s\mathbf{B} = -\tau\mathbf{N}. \quad (4)$$

With this definition, one will identify  $\mathbf{T}$ ,  $\mathbf{N}$ ,  $\mathbf{B}$  as the Frenet frame along the curve. The operator  $\nabla_s$  is  $\overline{\mathcal{F}}$ -linear and satisfies

$$\frac{\partial}{\partial s}\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \nabla_s\mathbf{X}, \mathbf{Y} \rangle + \langle \mathbf{X}, \nabla_s\mathbf{Y} \rangle \quad \forall \mathbf{X}, \mathbf{Y} \in \overline{\mathcal{X}}. \quad (5)$$

Let us then introduce two operators  $\delta_{\mathbf{X}}$  and  $\nabla_{\mathbf{X}}$ , which are determined by the usual variational calculus as in [3] with regarding  $\mathbf{X} \in \overline{\mathcal{X}}$  as a variational vector field. For  $\forall \mathbf{X} \in \overline{\mathcal{X}}$ , the  $\mathbf{R}$ -linear operator  $\delta_{\mathbf{X}}: \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}}$  is defined by

$$\delta_{\mathbf{X}}(fg) = (\delta_{\mathbf{X}}f)g + f(\delta_{\mathbf{X}}g) \quad \forall f, g \in \overline{\mathcal{A}}, \quad (6)$$

$$\delta_{\mathbf{X}}\partial^m = \partial^m\delta_{\mathbf{X}}, \quad \delta_{\mathbf{X}}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\delta_{\mathbf{X}} - (\lambda^{-1}\delta_{\mathbf{X}}\lambda)\frac{\partial}{\partial s}, \quad (7)$$

$$\delta_{\mathbf{X}}\lambda = \langle \lambda\mathbf{T}, \nabla_s\mathbf{X} \rangle, \quad (8)$$

$$\delta_{\mathbf{X}}\kappa = \langle \mathbf{N}, \nabla_s\nabla_s\mathbf{X} + c\mathbf{X} \rangle - \langle 2\kappa\mathbf{T}, \nabla_s\mathbf{X} \rangle, \quad (9)$$

$$\delta_{\mathbf{X}}\tau = \frac{\partial}{\partial s}\langle \kappa^{-1}\mathbf{B}, \nabla_s\nabla_s\mathbf{X} + c\mathbf{X} \rangle + \langle \kappa\mathbf{B}, \nabla_s\mathbf{X} \rangle - \langle \tau\mathbf{T}, \nabla_s\mathbf{X} \rangle. \quad (10)$$

And, for  $\forall \mathbf{X} \in \overline{\mathcal{X}}$ , the  $\mathbf{R}$ -linear operator  $\nabla_{\mathbf{X}}: \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$  is defined by

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\delta_{\mathbf{X}}f)\mathbf{Y} + f(\nabla_{\mathbf{X}}\mathbf{Y}) \quad \forall f \in \overline{\mathcal{A}}, \forall \mathbf{Y} \in \overline{\mathcal{X}}, \quad (11)$$

$$\nabla_{\mathbf{X}}\mathbf{T} = \langle \mathbf{N}, \nabla_s\mathbf{X} \rangle \mathbf{N} + \langle \mathbf{B}, \nabla_s\mathbf{X} \rangle \mathbf{B}, \quad (12)$$

$$\nabla_{\mathbf{X}}\mathbf{N} = \langle \kappa^{-1}\mathbf{B}, \nabla_s\nabla_s\mathbf{X} + c\mathbf{X} \rangle \mathbf{B} - \langle \mathbf{N}, \nabla_s\mathbf{X} \rangle \mathbf{T}, \quad (13)$$

$$\nabla_{\mathbf{X}}\mathbf{B} = -\langle \mathbf{B}, \nabla_s\mathbf{X} \rangle \mathbf{T} - \langle \kappa^{-1}\mathbf{B}, \nabla_s\nabla_s\mathbf{X} + c\mathbf{X} \rangle \mathbf{N}. \quad (14)$$

These operators satisfy

$$\delta_{\Phi\mathbf{X}} = \Phi\delta_{\mathbf{X}}, \quad \nabla_{\Phi\mathbf{X}} = \Phi\nabla_{\mathbf{X}} \quad \forall \Phi \in \overline{\mathcal{F}}, \forall \mathbf{X} \in \overline{\mathcal{X}}, \quad (15)$$

$$\delta_{\mathbf{X}}\langle \mathbf{Y}_1, \mathbf{Y}_2 \rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}_1, \mathbf{Y}_2 \rangle + \langle \mathbf{Y}_1, \nabla_{\mathbf{X}}\mathbf{Y}_2 \rangle \quad \forall \mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2 \in \overline{\mathcal{X}}. \quad (16)$$

**THEOREM 1** *The vector space  $\overline{\mathcal{X}}$  admits a Lie algebra structure*

$$[\mathbf{X}, \mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} \quad \forall \mathbf{X}, \mathbf{Y} \in \overline{\mathcal{X}}. \quad (17)$$

The algebra  $\overline{\mathcal{A}}$  is an  $\overline{\mathcal{X}}$ -module with the action  $\delta_X$ ,  $X \in \overline{\mathcal{X}}$ , namely,

$$(\delta_X \delta_Y - \delta_Y \delta_X - \delta_{[X, Y]})f = 0 \quad \forall X, Y \in \overline{\mathcal{X}}, \forall f \in \overline{\mathcal{A}}. \quad (18)$$

This is proved through the following step: First, show the equation (18) for  $f = \lambda, \kappa, \tau$  and then extend (18) to the whole  $\overline{\mathcal{A}}$ . Second, verify the Jacobi identity in  $\overline{\mathcal{X}}$  with the help of (18). Actual task is somewhat tedious but straightforward. It is apparent that the algebra  $\overline{\mathcal{F}}$  is an  $\overline{\mathcal{X}}$ -submodule of  $\overline{\mathcal{A}}$ , since  $\int$  commutes with  $\delta_X$ .

As was mentioned, the objects given above are related to a parametrized curve. Now we will fix most of the ambiguity of parametrization by imposing the relation  $\frac{\partial}{\partial s} \lambda = 0$ , which means that the parameter coincides with a multiple of the arclength. As a consequence, the velocity will be eliminated except for its zero-frequency component  $\ell := \int \lambda$ , the circumferential length along the closed curve.

Let  $\overline{\mathcal{I}}$  be the minimal ideal of  $\partial^{\mathbb{Z}}$ -algebra  $\overline{\mathcal{A}}$  such that  $\frac{\partial}{\partial s} \lambda \in \overline{\mathcal{I}}$ . Let  $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{I}}$ ,  $\mathcal{F} := \int(\mathcal{A})$ ,  $\mathcal{X} := \mathcal{A}N \oplus \mathcal{A}B$ ,  $\overline{\mathcal{I}}\overline{\mathcal{X}} := \overline{\mathcal{I}}T \oplus \overline{\mathcal{I}}N \oplus \overline{\mathcal{I}}B$ , and  $\overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}} := \mathcal{A}T \oplus \mathcal{A}N \oplus \mathcal{A}B$ .

There is no actual difference between the length  $\ell$  and the velocity  $\lambda$  in  $\mathcal{F}$  or in  $\mathcal{A}$ , because  $\frac{\partial}{\partial s} \lambda = 0$ . However, we use both symbols; we prefer denoting it by  $\ell$  if and only if we consider it as an element of  $\mathcal{F}$  rather than of  $\mathcal{A}$ .

Let  $\varphi$  be the surjection  $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}/\overline{\mathcal{A}}T$  or surjection  $\overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}} \rightarrow \mathcal{X}$ ,

$$\varphi(fT + gN + hB) := gN + hB \quad (19)$$

and let  $\wp$  be the injection  $\overline{\mathcal{F}} \times (\overline{\mathcal{X}}/\overline{\mathcal{A}}T) \rightarrow \overline{\mathcal{X}}$  or injection  $\mathcal{F} \times \mathcal{X} \rightarrow \overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}}$ ,

$$\wp(\Phi, gN + hB) := (\partial^{-1}(\lambda\kappa g) + \Phi)T + gN + hB. \quad (20)$$

The image  $\overline{\mathcal{X}}' := \wp(\overline{\mathcal{F}}, \overline{\mathcal{X}}/\overline{\mathcal{A}}T)$  of  $\wp$  preserves  $\overline{\mathcal{I}}$ , i.e.,  $\delta_{X'} \overline{\mathcal{I}} \subset \overline{\mathcal{I}} \forall X' \in \overline{\mathcal{X}}'$ , and is a Lie subalgebra of  $\overline{\mathcal{X}}$ . Hence  $\mathcal{A}$  is an  $\overline{\mathcal{X}}'$ -module, so is  $\mathcal{F}$ .

Recalling the definition of  $\nabla_s$ , we see that  $\nabla_s$  preserves  $\overline{\mathcal{I}}\overline{\mathcal{X}}$ , i.e.,  $\nabla_s(\overline{\mathcal{I}}\overline{\mathcal{X}}) \subset \overline{\mathcal{I}}\overline{\mathcal{X}}$ , hence makes sense as a map  $\overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}}$ . Similarly, we see that  $\delta_{X'}$  and  $\nabla_{X'}$  make sense as, respectively, a map  $\mathcal{A} \rightarrow \mathcal{A}$  and a map  $\overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}}$  if  $X'$  belongs to  $\mathcal{X}' := \wp(\mathcal{F}, \mathcal{X})$ . Formulae (3)–(16) hold also for these quotient spaces with the restriction  $X' \in \mathcal{X}'$  on  $X'$  of  $\delta_{X'}$  and  $\nabla_{X'}$ .

As a corresponding result for these quotient spaces, the image  $\mathcal{X}'$  of  $\wp$  in  $\overline{\mathcal{X}}/\overline{\mathcal{I}}\overline{\mathcal{X}}$  is a quotient Lie algebra of  $\overline{\mathcal{X}}'$ ,  $\partial^{\mathbb{Z}}$ -algebra  $\mathcal{A}$  is an  $\mathcal{X}'$ -module and the subalgebra  $\mathcal{F}$  is an  $\mathcal{X}'$ -submodule.

Furthermore, it is not difficult to show that  $\mathcal{F}T$  is an ideal of Lie algebra  $\mathcal{X}'$  and that  $\delta_T f = \frac{\partial}{\partial s} f \forall f \in \mathcal{A}$ , thus,

**PROPOSITION 2** *The vector space  $\mathcal{X}$  admits a Lie algebra structure*

$$[X, Y] := \varphi(\nabla_{X'} Y' - \nabla_{Y'} X'), \quad X' := \varphi(0, X), \quad Y' := \varphi(0, Y) \quad \forall X, Y \in \mathcal{X}. \quad (21)$$

*The algebra  $\mathcal{F}$  is an  $\mathcal{X}$ -module with the action  $\delta_{\varphi(0, X)}$  of  $X \in \mathcal{X}$ .*

We call an element of  $\mathcal{F}$  a function and an element of  $\mathcal{X}$  a vector field. Following the standard notation of differential geometry, we denote the action of vector field  $X$  to the function  $\Phi$  by the left action  $X\Phi := \delta_{\varphi(0, X)}\Phi$ . Since  $\mathcal{X}$  is an  $\mathcal{A}$ -vector space, function  $\Phi \in \mathcal{F} \subset \mathcal{A}$  can naturally act to the vector field  $X$ ; we denote it by the left action  $\Phi X$ .

As we have defined or shown,  $\mathcal{F}$  is a commutative and associative algebra,  $\mathcal{X}$  is a Lie algebra,  $\mathcal{F}$  is a left  $\mathcal{X}$ -module and  $\mathcal{X}$  is a left  $\mathcal{F}$ -module. It is obvious that each vector field  $X \in \mathcal{X}$  acts as a derivation to the product either  $\Phi\Psi$  or  $\Phi Y \forall \Phi, \Psi \in \mathcal{F}, \forall Y \in \mathcal{X}$ . In addition,  $(\Phi X)\Psi = \Phi(X\Psi)$  holds for  $\forall \Phi, \Psi \in \mathcal{F}, \forall X \in \mathcal{X}$ . These are condition enough to introduce, as in Sect. 4, the differential calculus.

Let us introduce a Riemannian metric for  $\mathcal{X}$ . Although we have no proof, it seems true that

$$\int f g = 0 \quad \forall g \in \mathcal{A} \Rightarrow f = 0 \quad (22)$$

for  $\forall f \in \mathcal{A}$ . Absense of the proof is not so serious, because it is at least possible to impose this rule; if (22) is false, we can redefine  $\mathcal{A}$  (and accordingly  $\mathcal{F}, \mathcal{X}$ ) to turn (22) into true with preserving the  $\partial^{\mathbf{Z}}$ -algebra structure (*cf.* Example 2) and, in addition, operators introduced in this section make sense for these redefined objects. Therefore, in particular, the inner product  $( , ) : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{F}$ ,

$$(g_1 N + h_1 B, g_2 N + h_2 B) = \int \lambda(g_1 g_2 + h_1 h_2) \quad \forall g_1, g_2, h_1, h_2 \in \mathcal{A} \quad (23)$$

is supposed non-degenerate and is thought of as a Riemannian structure. Then, for given any function  $\Phi$ , there uniquely exists a vector field  $X$  (the gradient of  $\Phi$ ) such that  $(X, Y) = Y\Phi \forall Y \in \mathcal{X}$ .

The parametrization of the curve have been fixed except for the ambiguity of rigid slide by imposing the relation  $\frac{\partial}{\partial s}\lambda = 0$ . And there are no elements in  $\mathcal{F}$  (and even in  $\overline{\mathcal{F}}$ ) that can distinguish the difference within this ambiguity. The objects  $\mathcal{F}$  and  $\mathcal{X}$  can, hence, be thought of as the algebra of functions and Lie algebra of vector fields, respectively, on (a certain quotient space of) the space consisting of the points represented by an unparametrized closed curve with non-vanishing length  $\ell$  and non-vanishing curvature  $\kappa$  living in a 3-dimensional space  $M$  of constant curvature equal to  $c$ . We denote the set of these algebraic objects by  $ECL(c)$  or simply by  $ECL$ . We use phrases as if  $ECL$  denoted a Riemannian manifold.

## 4 A Structure Related to the Integrability

We denote by  $\otimes$  the operation making  $\mathcal{F}$ -tensor product. Let  $\mathcal{D}^p$  denote the vector space of  $\mathcal{F}$ -linear maps  $\omega: \mathcal{X}^{\otimes p} \rightarrow \mathcal{F}$  such that  $f := \omega(X_1 \otimes \cdots \otimes X_p)$ ,  $X_i \in \mathcal{X}$ , is skew-symmetric, if  $p \geq 2$ , under the exchange of  $X_i$  and  $X_j$ ,  $i \neq j$ , and has algebraic expression  $f = f(\cdots)$  written with the ingredients  $g_i, h_i$  ( $g_i N + h_i B := X_i$ ),  $\omega$ -dependent elements of  $\mathcal{A}$  and operators  $\partial^m$ . Such an element  $\omega$  of  $\mathcal{D}^p$  is called a  $p$ -form.

The exterior derivative  $d: \mathcal{D}^p \rightarrow \mathcal{D}^{p+1}$  and the exterior product  $\wedge: \mathcal{D}^p \times \mathcal{D}^q \rightarrow \mathcal{D}^{p+q}$  are introduced algebraically in the usual manner. To fix the convention, we present three formulae  $(d\Phi)(X) = X\Phi$ ,  $(d\omega)(X \otimes Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$  and  $(\omega \wedge \eta)(X \otimes Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X) \forall X, Y \in \mathcal{X}, \forall \Phi \in \mathcal{D}^0 = \mathcal{F}, \forall \omega, \eta \in \mathcal{D}^1$ .

Given 1-form  $\xi \in \mathcal{D}^1$ , we associate a vector field  ${}^*\xi \in \mathcal{X}$  by  $({}^*\xi, Y) = \xi(Y) \forall Y \in \mathcal{X}$  with the Riemannian structure (23). Given operator  $H: \mathcal{X} \rightarrow \mathcal{X}$ , we associate an operator  $H^*: \mathcal{D}^1 \rightarrow \mathcal{X}$  by  $(H^*)\xi = H({}^*\xi) \forall \xi \in \mathcal{D}^1$ . Operator  $H: \mathcal{X} \rightarrow \mathcal{X}$  is called a skew-adjoint operator if  $H$  is  $\mathcal{F}$ -linear and obeys  $(X, HY) = -(HX, Y) \forall X, Y \in \mathcal{X}$ .

Here, we briefly review geometrical notions and statements regarding (bi-)Hamiltonian structure [6], see also [7]. Given two skew-adjoint operators  $H_1$  and  $H_2$ , define the 3-form  $[H_1, H_2] \in \mathcal{D}^3$ ,

$$\begin{aligned} & [H_1, H_2](X \otimes Y \otimes Z) \\ &= \left\{ (H_1 X)(H_2 Y, Z) + (Z, [H_1 Y, H_2 X]) + (H_1 \leftrightarrow H_2) \right\} + \text{cycle}(X, Y, Z) \quad (24) \\ & \qquad \qquad \qquad \forall X, Y, Z \in \mathcal{X}. \end{aligned}$$

The 3-form  $[H_1, H_2]$  is the dual (for notational simplicity) of the Schouten bracket between  $H_1^*$  and  $H_2^*$ . A skew-adjoint operator  $J$  (or  $J^*$  [6]) is called a Hamiltonian operator if  $[J, J] = 0$ . A Hamiltonian operator  $J$  induces a Poisson structure

$$\{\Phi, \Psi\} = (J^* d\Phi)\Psi \quad \forall \Phi, \Psi \in \mathcal{F} \quad (25)$$

and  $J^* d$  provides a morphism of Lie algebras  $\mathcal{F} \rightarrow \mathcal{X}$ . A pair of two Hamiltonian operators  $J$  and  $H$  such that  $[J, H] = 0$  is called a bi-Hamiltonian structure (or Hamiltonian pair [6]), which implies, with certain further conditions, the existence of a sequence of functions in involution or Poisson-commutative functions.

Let us return to the problem of *ECL*. Define operator  $J: \mathcal{X} \rightarrow \mathcal{X}$ ,

$$J(gN + hB) = hN - gB \quad \forall h, g \in \mathcal{A} \quad (26)$$

and operator  $K: \mathcal{X} \rightarrow \mathcal{X}$ ,

$$K(X) = J \varphi \nabla_s \wp(0, JX) \quad \forall X \in \mathcal{X}. \quad (27)$$

It is easy to show that the operators  $J$  and  $K$  are skew-adjoint. These are operators for  $ECL$  analogous to those for  $BAL$  given in [3], where J. Langer and R. Perline showed that the operators corresponding to  $J$  and  $K$  above generate recursively the sequense of vector fields on  $BAL$  associated with certain functions in involution. Below, we will consider a recursively generated sequense of 1-forms on  $ECL$  defined analogously.

Let  $\xi_n$ ,  $n = 0, 1, \dots$  be the sequense of 1-forms defined by

$$\xi_n := \ell^{n-1} d\ell \circ (KJ^{-1})^n, \quad i.e., {}^*\xi_n := \ell^{n-1} (J^{-1}K)^n (-\kappa N). \quad (28)$$

**THEOREM 3** *The dual (24) of the Schouten brackets for skew-adjoint operators  $J$  and  $K$  defined by (26) and (27), respectively, are*

$$[J, J](X \otimes Y \otimes Z) = 0, \quad (29)$$

$$[J, K](X \otimes Y \otimes Z) = 2\xi_0(JX)(Y, KZ) + \text{cycle}(X, Y, Z), \quad (30)$$

$$[K, K](X \otimes Y \otimes Z) = 2\xi_0(JX)(Y, (3KJ^{-1}K - cJ)Z) + \text{cycle}(X, Y, Z) \quad (31)$$

with  $\xi_0$  of (28).

The proof we know is not so easy nor so straightforward. However, we would like to omit presenting it, since it requires a tedious calculation and no tricky technics.

The theorem above says  $J$  is a Hamiltonian operator but the pair  $J$  and  $K$  is not of a bi-Hamiltonian. It is implied by investigations of [2, 11] that  $J$  is a Hamiltonian operator. The vortex filament equation is expressed by the Hamiltonian function  $\ell$  with the Poisson bracket induced from  $J$  [2, 11].

**THEOREM 4** *The sequense of 1-forms  $\xi_n$  defined by (28) satisfies*

$$d\xi_n = \sum_{k \in \mathbf{Z}, 0 < 2k < n} (2k - n)(\xi_k \wedge \xi_{n-k} - c\ell^2 \xi_{k-1} \wedge \xi_{n-k-1}). \quad (32)$$

Before sketching the proof, we note that the equation

$$\begin{aligned} & 2 \left( d\xi \circ (H_1 \otimes H_1) - d\eta \circ (H_1 \otimes H_2 + H_2 \otimes H_1) + d\zeta \circ (H_2 \otimes H_2) \right) (X \otimes Y) \\ &= [H_1, H_1]({}^*\xi \otimes X \otimes Y) - 2[H_1, H_2]({}^*\eta \otimes X \otimes Y) + [H_2, H_2]({}^*\zeta \otimes X \otimes Y) \quad (33) \\ & \qquad \qquad \qquad \forall X, Y \in \mathcal{X} \end{aligned}$$

is fulfilled for any skew-adjoint operators  $H_1$  and  $H_2$  and 1-forms  $\xi, \eta, \zeta$  whenever

$$H_1{}^*\xi = H_2{}^*\eta, \quad H_1{}^*\eta = H_2{}^*\zeta. \quad (34)$$

Taking  $H_1 = J$ ,  $H_2 = K$ ,  $\xi = \ell^{2-n}\xi_{n+2}$ ,  $\eta = \ell^{1-n}\xi_{n+1}$  and  $\zeta = \ell^{-n}\xi_n$  in (33) and using Theorem 3, we can derive a recursive relation for  $d\xi_k$ . The recursive relation is simplified by

$$\xi_i(J^*\xi_j) = 0, \quad \xi_i(K^*\xi_j) = 0, \quad (35)$$

which follow from skew-adjointness of  $J$  and  $K$ . Using this recursive relation, one can prove Theorem 4 by induction with the initial data  $d\xi_0 = d\xi_1 = 0$ , which are easily seen from  $\xi_0 = \ell^{-1}d\ell$  and  $\xi_1 = d\int(-\lambda\tau)$ .

As we will see below, the relation (32) is to strongly suggest the integrability.

**PROPOSITION 5** *Suppose the triviality of the de Rham cohomology of degree 1, i.e., suppose every closed 1-form is exact. Let a function  $c\ell^2$  and 1-forms  $\xi_0, \dots, \xi_m$  be given. If these satisfy (32) and  $d(c\ell^2) = 2c\ell^2\xi_0$ , then there exist functions  $f_1, \dots, f_m$  such that*

$$df_n = \xi_n + \sum_{k \in \mathbf{Z}, 1 < k < n} (1-k)(f_k\xi_{n-k} - c\ell^2 f_{k-1}\xi_{n-k-1}), \quad 1 \leq n \leq m. \quad (36)$$

This is easily proved by induction in  $m$ . The proposition is of general character; it is not necessary to suppose that the differential calculus is that on  $ECL$ .

If there exist functions  $f_1, f_2, \dots \in \mathcal{F}$  satisfying (36) for  $\xi_n$  of (28), it is readily seen from (35) and  $\mathcal{F}$ -linearity of  $J$  that these functions and  $\ell$  are in involution. Moreover, it is not difficult to show that  $\xi_0, \xi_1, \dots$  are linearly independent over  $\mathcal{F}$ , so are the flows associated, via  $J^*d$ , with the functions  $\ell, f_1, f_2, \dots$  (if exist).

We do not know whether the de Rham cohomology of degree 1 for  $ECL$  is trivial or not. Therefore, the existence of these functions on  $ECL$  is, rigorously speaking, not proved. However, their existence itself seems true; we conjecture that

$$f_n = \ell^{n-1} \sum_{k \in \mathbf{Z}, 0 \leq 2k \leq n} \frac{(2k)!}{2^{2k}(k!)^2} c^k \int \lambda P_{n-2k} \quad (37)$$

with (1) solve the equation (36), cf. Sect. 5 for the case  $c = 0$ .

We close this section with the following remarks. Recalling the argument above, we pick out the set  $(J, K, \ell)$  of invertible Hamiltonian operator  $J$ , skew-adjoint operator  $K$  and non-vanishing function  $\ell$  as a geometrical structure of  $ECL$ . Theorem 3 and the equation  $d(d\ell \circ (KJ^{-1})) = 0$  are, then, essential for the integrability. This structure is enough to define recursively a sequense of 1-forms, whose external derivatives obey the relation (32), and suggests the existence of the sequense of functions in involution. This seems to indicate a possibility of generalizing bi-Hamiltonian structure into something relaxed but preserving the nature supporting the integrability.

## 5 A Remark on Langer-Parline's Formula

Let us introduce a  $\mathbf{Z}$ -grading to  $\overline{\mathcal{A}}, \overline{\mathcal{F}}, \overline{\mathcal{X}}, \mathcal{A}, \mathcal{F}, \mathcal{X}$  and  $\mathcal{D}^p$  as follows. Let  $\deg \mathbf{R} = 0$ ,  $\deg \lambda = -1$ ,  $\deg \kappa = \deg \tau = 1$ ,  $\deg c = 2$  (with regarding  $c$  as an indeterminate),  $\deg \mathsf{T} = \deg \mathsf{N} = \deg \mathsf{B} = 0$ . Let operators/maps  $\partial^m, \int$  and  $\varphi$  do not change the degree. This grading is compatible with operators  $\delta_X, \nabla_X$ , etc. It is easily shown that the external derivative  $d$  increases the degree by 1. This is used below.

As was mentioned in Sect. 1, results on *BAL* regarding local polynomials can be translated into the statements for *ECL*. We will do so, but we would like to note that this relies on the assumptions (i) the convergence of improper integrals posed in [3] and (ii) harmlessness of reading  $\kappa^{(i)}$  and  $\tau^{(i)}$ , which in [3] are variables depending on the curve of *BAL*, as indeterminates.

Below we set  $c = 0$  for simplicity, but (41) can be derived even if  $c \neq 0$  with the help of a hint in [9].

We can read (A)–(C) in [3]: Let  $X_{n-2} := J^* d \int \lambda P_n \in \mathcal{X}$  with  $P_n$  of (1), where (A)  $P_n$  are local polynomials. (B) There exist local polynomials  $P_{\mathsf{T}, n}, P_{\mathsf{N}, n}$  and  $P_{\mathsf{B}, n}$  such that  $X_{n-2} = P_{\mathsf{N}, n} N + P_{\mathsf{B}, n} B$  and  $\frac{\partial}{\partial s} P_{\mathsf{T}, n} = \kappa P_{\mathsf{N}, n}$ . We see  $\deg P_n = \deg P_{\mathsf{T}, n-1} = n$  with fixing the ambiguity of  $P_{\mathsf{T}, n}$  appropriately. Then, (C)

$$X_{(n+1)-2} = J \varphi \nabla_s (P_{\mathsf{T}, n} \mathsf{T} + P_{\mathsf{N}, n} \mathsf{N} + P_{\mathsf{B}, n} \mathsf{B}). \quad (38)$$

Note that the right hand side slightly differs from  $KJ^{-1} X_{n-2}$  with  $J$  of (26) and  $K$  of (27).

From these translated facts, we see  $KJ^{-1} X_{n-2} - X_{(n+1)-2} = (\int P_{\mathsf{T}, n}) J(-\kappa N)$ , which results in

$$\zeta_n = d \int \lambda P_n + \ell^{-1} \sum_{k \in \mathbf{Z}, 2 \leq k \leq n} (\int \lambda P_{\mathsf{T}, k-1}) \zeta_{n-k}, \quad (39)$$

where we have put  $\zeta_n := \ell^{1-n} \xi_n$ . Using this relation and (32) with  $c = 0$ , we can show that

$$\sum_{k \in \mathbf{Z}, 2 \leq k \leq n} d \Delta_k \wedge \zeta_{n-k} = 0, \quad \Delta_k := \int \lambda((k-1) P_k - P_{\mathsf{T}, k-1}). \quad (40)$$

This equation for  $n = 2$  implies that there exists  $\Phi \in \mathcal{F}$  such that  $d \Delta_2 = \Phi \zeta_0$ . We note that there exist local polynomials  $P_{\mathsf{N}}$  and  $P_{\mathsf{B}}$  and some integer  $m$  such that  $*d \int \lambda P = \kappa^{-m} (P_{\mathsf{N}} N + P_{\mathsf{B}} B)$  for every local polynomial  $P$ . Since  $*\zeta_0 = -\kappa N$ , we see  $\Phi \kappa^{m+1}$  for some integer  $m$  is also a local polynomial, while  $\Phi \in \mathcal{F}$ , hence  $\Phi \in \mathbf{R}$ . We now have two ways of evaluating  $\deg \Phi$ , namely,  $\deg \Phi = 1 + \deg \Delta_2 - \deg \zeta_0 = 2$  and  $\deg \Phi = \deg(\mathbf{R}) = 0$ . This mismatch is permissible only if  $\Phi = 0$ , hence  $d \Delta_2 = 0$ . The argument can be extended iteratively to assert  $d \Delta_3 = d \Delta_4 = \dots = 0$ , i.e.,

$$(n-1) \int \lambda P_n \equiv \int \lambda P_{\mathsf{T}, n-1}, \quad n = 2, 3, \dots, \quad (41)$$

where  $\equiv$  denotes the equivalence in the image in  $\mathcal{D}^1$  of  $d$ . This proves significant part of the Langer-Perline's inspection [3], which says the validity even in  $\mathcal{F}$  of the formula (41).

## Acknowledgements

The author is grateful to Dr. Y. Yasui for profitable discussions.

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